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CLASSIFICATION OF KILLING-TRANSVERSALLY SYMMETRIC SPACES

In memory of Professor Kentaro Yano

By

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Abstract. We treat Killing-transversally symmetric spaces (briefly, KTS-spaces), that is, Riemannian manifolds equipped with a complete unit Killing vector field such that the reflections with respect to the flow lines of that field can be extended to global isometries. Such manifolds are homogeneous spaces equipped with a naturally reductive homogeneous structure and they provide a rich set of examples of reflection spaces. We prove that each simply connected reducible KTS-space M is a Riemannian product of a symmetric space M' and a special kind of KTS-space M'', called a contact KTS-space. Such a particular manifold M'' is an irreducible, odd-dimensional principal G^1-bundle over a Hermitian symmetric space. The main purpose of the paper is to give a classification of this special class of manifolds M''.

1. Introduction

A Riemannian manifold (M,g) is said to be a *locally Killing-transversally symmetric space* (briefly, a *locally* KTS-*space*) if it is equipped with a Riemannian foliation whose leaves are generated by a unit Killing vector field ξ and such that the local reflections with respect to these leaves are isometries. It is said to be a *globally* KTS-space if ξ is complete and if the local isometric reflections can be extended to global ones. Several aspects of the geometry of these spaces have been treated in [7]–[13] where a lot of examples are given.

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The motivation for the consideration of this class of Riemannian manifolds stems from the study of two particular classes of manifolds. On the one hand, these spaces form a subclass of the class of manifolds equipped with a *transversally symmetric Riemannian foliation* [27], [28] and on the other hand, they extend the class of (*locally* or *globally*) *φ-symmetric spaces*. The latter ones are considered in contact geometry [25], [30] where they play a similar role as the Hermitian symmetric spaces in Kähler geometry. Their classification is treated in [3], [4], [15], [20]. (See also [6] for a survey, more details and further references.)

It is worthwhile to note that the globally KTS-spaces provide a large class of examples of *reflection spaces*. The latter ones were introduced and studied in [21].

The main purpose of this paper is to discuss the classification of the (globally) KTS-spaces. Such a space is necessarily a homogeneous space endowed with a naturally reductive homogeneous structure (see Section 2). In Section 5 we prove that a reducible simply connected KTS-space M is a Riemannian product of a Riemannian symmetric space M' and a KTS-space M'' of a special type, namely such that the dual one-form η of ξ is a contact form on M''. Such spaces M'' are called *contact* KTS-spaces. They are always irreducible and odd-dimensional. In Section 2 we show that KTS-spaces are principal G^1-bundles over a symmetric space. In Section 3 we prove that for a simply connected contact KTS-space the base space of the fibration is always a Hermitian symmetric space. This fact is used in Section 4 where we treat the construction and the classification of the contact KTS-spaces in detail. Section 2 also contains some preliminary material.

2. Locally and globally Killing-transversally symmetric spaces

Let (M, g) be an n-dimensional, smooth, connected Riemannian manifold with $n \geq 2$. ∇ denotes the Levi Civita connection and R the associated Riemannian curvature tensor defined by

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for all $U, V \in \mathfrak{X}(M)$, the Lie algebra of C^∞ vector fields on M.

Further, let ξ be a unit Killing vector field on (M,g) and $\mathfrak{F}_\xi$ the flow generated by it. It is a Riemannian flow which is called an *isometric* flow [26]. The leaves of this Riemannian foliation are geodesics and moreover, a geodesic which is orthogonal to the flow field ξ at one of its points is orthogonal to it at all of its points. Such geodesics are called *transversal* (or *horizontal*) geodesics.

Further, the foliation is locally a Riemannian submersion. So, let $m \in (M,g)$ and let $\mathcal{U}$ be a small open neighborhood of m such that ξ is regular on $\mathcal{U}$. Then the map $\pi : \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$ is a submersion. Let g' denote the metric on $\mathcal{U}$ defined by

$$g'(X',Y') = g(X'^*,Y'^*)$$

for $X',Y' \in \mathfrak{X}(\mathcal{U}')$ where X'^*,Y'^* denote the horizontal lifts of X',Y' with respect to the $(n-1)$-dimensional horizontal distribution of $\mathcal{U}'$ determined by the one-form η on M given by

$$\eta(U) = g(U,\xi),$$

$U \in \mathfrak{X}(M)$. Then the Levi Civita connection ∇' of g' is determined by

$$\nabla'_{X'}Y' = \pi_* \nabla_{X'^*} Y'^*,$$

$X',Y' \in \mathfrak{X}(\mathcal{U}')$.

Next, we may use the O'Neill tensors A and T to study the flow $\mathfrak{F}_\xi$. See [22] for more detrails (and also [1], [24], [26]). In our case $T = 0$ since the leaves are geodesics. For the integrability tensor A we have

$$A_U\xi = \nabla_U\xi, \; A_\xi U = 0,$$
$$A_XY = (\nabla_XY)^v = -A_YX, \; g(A_XY,\xi) = -g(A_X\xi,Y)$$

where $U \in \mathfrak{X}(M)$ and X,Y are horizontal vectors fields. v denotes the vertical component. Further, put

$$HU = -A_U\xi$$

and define the $(0,2)$-tensor field h by

$$h(U,V) = g(HU,V),$$

$U,V \in \mathfrak{X}(M)$. Since ξ is a Killing field, h is clearly skew-symmetric. Then we obtain

$$A_XY = h(X,Y)\xi = \frac{1}{2}\eta([X,Y])\xi$$

for all horizontal fields X, Y. This yields

$$h = -d\eta. \tag{2.1}$$

Here we note that $A = 0$, or equivalently $h = 0$, if and only if the horizontal distribution is integrable and in that case, since $T = 0$, (M,g) is locally a product of an $(n-1)$-dimensional manifold and a line. Further, ∇ and ∇' are related by

(2.2) $$\nabla'_{X'^*}Y'^* = (\nabla'_{X'}Y')^* + h(X'^*, Y'^*)\xi.$$

Using the given formulas we obtain easily

LEMMA 2.1. *We have*

$$(\nabla_\xi h)(X,Y) = g((\nabla_\xi A)_X Y, \xi) = 0,$$
$$R(X,Y,Z,\xi) = (\nabla_Z h)(X,Y),$$
$$R(X,\xi,Y,\xi) = g(HX,HY) = -g(H^2X,Y)$$

for all horizontal X, Y, Z.

Since $H\xi = 0$, it follows that the sectional curvature $K(X,\xi) = 0$ for all horizontal X if and only if $h = 0$. Moreover, $K(X,\xi) > 0$ for all X if and only if H has maximal rank $n-1$ in which case n is necessarily odd. This is equivalent to the statement that η is a contact form. A flow $\mathfrak{F}_\xi$ with this property is called a *contact* flow and a flow $\mathfrak{F}_\xi$ such that $R(X,Y,Z,\xi) = 0$ for all horizontal X, Y, Z is said to be a *normal* flow [8], [12].

From Lemma 2.1 we then obtain

PROPOSITION 2.1. *Let $\mathfrak{F}_\xi$ be an isometric flow on (M,g). Then $\mathfrak{F}_\xi$ is normal if and only if*

$$(\nabla_U H)V = g(HU,HV)\xi + \eta(V)H^2U$$

for all $U, V \in \mathfrak{X}(M)$.

Further, for a normal flow the curvature tensor satisfies the identities

$$R_{UV}\xi = \eta(V)H^2U - \eta(U)H^2V,$$
$$R_{U\xi}V = g(HU,HV)\xi + \eta(V)H^2U,$$

$U, V \in \mathfrak{X}(M)$. This and (2.2) yield that the curvature tensors of ∇ and ∇' are related by

(2.3) $$(R'_{X'Y'}Z')^* = R_{X'^*,\vartheta Y'^*}Z'^* - g(HY'^*, Z'^*)HX'^* + g(HX'^*, Z'^*)HY'^* + 2g(HX'^*, Y'^*)HZ'^*,$$

for all $X', Y', Z' \in \mathfrak{X}(\mathcal{U}')$ [8].

Now, let H' be the tensor field of type (1,1) on $\mathcal{U}'$ determined by $H'X' = \pi_*(HX'^*)$, $X' \in \mathfrak{X}(\mathcal{U}')$. Then H' is skew-symmetric with respect to g'

and we have

$$(\nabla_{X'^*}H)Y'^* = ((\nabla'_{X'}H')Y')^* + g(HX'^*, HY'^*)\xi.$$

Using this and Proposition 2.1 we derive that $\mathfrak{F}_\xi$ is normal if and only if $\nabla'H' = 0$ (see also [8]) and then we have

(2.4) $$R'_{H'X'Y'} + R'_{X'H'Y'} = 0$$

for $X', Y' \in \mathfrak{X}(\mathcal{U}')$.

Next, let (M,g) be equipped with an isometric flow $\mathfrak{F}_\xi$ and define the tensor field T of type $(1,2)$ by

(2.5) $$T_U V = g(HU, V)\xi + \eta(U)HV - \eta(V)HU$$

for all tangent U,V. (Note that this T is unrelated to the O'Neill tensor.) Then we have $T_U U = 0$. Moreover, $\overline{\nabla} = \nabla - T$ defines a metric connection which is called the *canonical connection* of the isometric flow $\mathfrak{F}_\xi$ [8]. Its torsion $\overline{K}$ is given by $\overline{K} = -2T$ and its geodesics are the same as those of ∇. A direct computation shows that ξ and η are $\overline{\nabla}$-parallel. Further, $\mathfrak{F}_\xi$ is normal if and only if $\overline{\nabla}T = 0$ [8], [12] and in this case we have

(2.6) $$\overline{R}_{UV} = R_{UV} + [T_U, T_V] - 2T_{T_U V}.$$

In what follows we define now the locally and globally Killing-transversally symmetric spaces. Therefore, let $m \in M$ and let σ be the flow line of ξ through it. A local diffeomorphism s_m of M defined in a neighborhood $\mathcal{U}$ of m is said to be the *(local) reflection with respect to* σ if for every transversal geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathcal{U}$ and σ, we have

$$(s_m \circ \gamma)(s) = \gamma(-s)$$

for all s with $\gamma(\pm s) \in \mathcal{U}$, s being the arc length of γ. Then $S_m = s_{m_*}(m)$ is given by

$$S_m = (-I + 2\eta \otimes \xi)(m)$$

and it is a linear isometry. Moreover, s_m satisfies

$$s_m = \exp_m \circ S_m \circ \exp_m^{-1}.$$

First, we state

DEFINITION 2.1. *A locally Killing-transversally symmetric space* (briefly a *locally* KTS-*space*) is a Riemannian manifold (M,g) equipped with an isometric flow $\mathfrak{F}_\xi$ such that the local reflection s_m with respect to the flow line through it is a (local) isometry for all $m \in M$. In what follows we will denote $(M, g, \mathfrak{F}_\xi)$ by (M, g, ξ).

For the locally KTS-spaces we have the following characterizations by using the curvature tensor R and the canonical connection.

PROPOSITION 2.2 [8]. *The following statements are equivalent for an* (M, g, ξ)*:*

(i) (M, g, ξ) *is a locally* KTS-*space;*

(ii) $\mathfrak{F}_\xi$ is normal and $(\nabla_X R)(X, Y, X, Y) = 0$ for all horizontal X, Y;

(iii) $\overline{\nabla} R = \overline{\nabla} H = 0$ *(or equivalently,* $\overline{\nabla} \overline{R} = \overline{\nabla} H = 0$*).*

PROPOSITION 2.3 [10]. *Let* $\mathfrak{F}_\xi$ *be a contact flow on* (M,g)*. Then* (M, g, ξ) *is a locally* KTS-*space if and only if* $\mathfrak{F}_\xi$ *is normal and*

$$(\nabla_X R)(X, HX, X, HX) = 0$$

for all horizontal X*.*

PROPOSITION 2.4 [8]. *Let* $\mathfrak{F}_\xi$ *be a normal flow on* (M,g)*. Then* (M, g, ξ) *is a locally* KTS-*space if and only if each base space* $\mathcal{U}'$ *of a local Riemannian submersion* $\pi : \mathcal{U} \to \mathcal{U}' = \mathcal{U} / \xi$ *is a locally symmetric space.*

Locally KTS-spaces are locally homogeneous spaces. More precisely, using the theory of homogeneous structures studied in [29], we have

PROPOSITION 2.5 [8]. (M, g, ξ) *is a locally* KTS-*space if and only if the tensor field* T *defines a homogeneous structure on it.*

Note that, because $T_U U = 0$, T determines then a *naturally reductive* structure [29].

Secondly, we consider

DEFINITION 2.2. Let (M,g) be a Riemannian manifold and ξ a nowhere vanishing complete Killing vector field on M. Then (M, g, ξ) is said to be a *(globally) Killing-transversally symmetric space* (briefly, a KTS-*space*) if and only if each local reflection s_m can be extended to a global isometry.

Note that this implies that ξ is a unit vector field.

Clearly, any KTS-space is a locally KTS-space. Further, we have

THEOREM 2.1 [7]. *A complete, connected, simply connected locally* KTS-*space is a* KTS-*space.*

From this we get at once

COROLLARY 2.1. *Let $\tilde{M}$ be the universal covering of a complete locally* KTS-*space (M,g,ξ) and Ψ the covering map. Then $(\tilde{M},\tilde{g}=\Psi^*g,\tilde{\xi})$, where $\tilde{\xi}$ is the lift of ξ, is a* KTS-*space.*

It is worthwhile to note here that the consideration of KTS-spaces was motivated by the study of *φ-symmetric spaces*. These spaces were introduced by Takahashi [25] in Sasakian geometry where they may be considered as the analogs of Hermitian symmetric spaces in Kähler geometry. These manifolds are Sasakian spaces with a characteristic vector field ξ which generates a one-parameter group of global isometries and such that the local reflections with respect to the integral curves of this field ξ can be extended to global automorphisms of the contact metric structure. Sasakian space forms are the most simple examples but there are a lot more. (We refer to [2], [32], [3], [6], [15] for more information and further references.) Further, they provide examples of KTS-spaces and moreover, if (M,g,ξ) is a φ-symmetric space, then also $(M,c^{-2}g,c\xi)$ is a KTS-space for any non-vanishing constant c. Conversely, it is proved in [8] that if (M,g,ξ) is a KTS-space with $K(X,\xi)=c^2\neq 0$ for all horizontal X, then $(M,c^2g,c^{-1}\xi)$ is equipped with a φ-symmetric structure having $c^{-1}\xi$ as characteristic vector field.

In the rest of the paper a lot of other examples will be given but we will now describe some easy ones. First, it is clear that any Euclidean space is a KTS-space by considering an arbitrary unit parallel vector field as Killing vector field. Further, all odd-dimensional spheres $S^{2k+1}(r)$ are also KTS-spaces. To see this, consider $S^{2k+1}(r)$ as a hypersphere in C^{k+1} and put $\xi=JN$ where N is a unit normal vector field of the hypersphere and J the natural complex structure of C^{k+1}. (See for example [2].) In this context we have

PROPOSITION 2.6 [7]. *Let (M,g) be an irreducible simply connected symmetric space and $\mathfrak{F}_\xi$ an isometric flow on it. Then (M,g,ξ) is a* KTS-*space if and only if (M,g) is isometric to an odd-dimensional sphere.*

Next, using Definition 2.2 we obtain the following list of examples:

(i) $(M_1\times M_2,\bar{g},(\xi,0))$ where (M_1,g_1,ξ) is a KTS-space, (M_2,g_2) a symmetric space and $\bar{g}$ the product metric. For $(m_1,m_2)\in M_1\times M_2$ the reflection is given by

$$(p_1, p_2) \mapsto (s^1_{m_1}(p_1), s^2_{m_2}(p_2))$$

where $s^1_{m_1}$ is the reflection with respect to the flow line of ξ through m_1 on M_1 and $s^2_{m_2}$ is the geodesic symmetry with center m_2 on M_2;

(ii) $(M \times \boldsymbol{R}, \bar{g}, (\xi, 0))$ and $(M \times S^1, \bar{g}, (\xi, 0))$ where (M, g, ξ) is a KTS-space;

(iii) $\left(M \times \boldsymbol{R}, \bar{g}, \left(0, \frac{d}{dt}\right)\right)$ and $(M \times S^1, \bar{g}, (0, \xi))$ where (M, g) is a symmetric and ξ a unit tangent vector field on S^1.

Now we shall focus our attention on the homogeneity of the KTS-spaces. Let $A(M)$ denote the group of all isometries of (M, g, ξ) leaving ξ invariant, that is, the automorphisms of the KTS-space (M, g, ξ). It is a closed subgroup of the full isometry group $\mathfrak{I}(M, g)$ and hence a Lie group. Since the reflections s_m preserve ξ for all $m \in M$, $s_m \in A(M)$. Further, let G^1 be the one-parameter group of global transformations ψ_t generated by the vector field ξ. Then G^1 is a Lie subgroup of $A(M)$ which belongs to its center. We have

THEOREM 2.2. *If (M, g, ξ) is a* KTS*-space, then $A(M)$ acts transitively on M.*

PROOF. Since the group of isometries generated by all global reflections with respect to the flow lines and also G^1 belong to $A(M)$, the proof is essentially the same as that given in [8] for the local version. ■

Using Proposition 2.5 and the note following it, we have

THEOREM 2.3. *A simply connected* KTS*-space is a naturally reductive homogeneous space.*

Since M is connected, the identity component $G = A_o(M)$ of $A(M)$ acts transitively on M too and M can be identified with the coset manifold G/K, where K is the isotropy subgroup of G at some point $o \in M$, under the diffeomorphism $gK \to g(o)$, $g \in G$. Let $(\tilde{G}, \tilde{\pi})$ be the universal covering of G. Then $\tilde{G}$ is a Lie group and $\tilde{\pi}: \tilde{G} \to G$ is a Lie group homomorphism. Denote by $\tilde{K}$ the identity component of $\tilde{\pi}^{-1}(K)$. Then $\tilde{G}/\tilde{K}$ is simply connected. The map $\Psi: \tilde{G}/\tilde{K} \to G/K$ given by $\Psi(\tilde{g}\tilde{K}) = \tilde{\pi}(\tilde{g})K$ is differentiable and onto. If on $\tilde{G}/\tilde{K}$ we consider the metric tensor $\Psi^* g$, then Ψ is a local isometry and $(\tilde{G}/\tilde{K}, \Psi)$ is a covering of M (see, for example, [14, p. 74]). So, $\tilde{M} = \tilde{G}/\tilde{K}$ and from Corollary 2.1 we get that $(\tilde{M} = \tilde{G}/\tilde{K}, \tilde{g}, \tilde{\xi})$ is a KTS-space. Note that, since $\tilde{\pi}_*$ is an isomorphism, $\tilde{K}$ coincides with the connected subgroup of $\tilde{G}$ associated with the Lie algebra $\mathfrak{k}$ of K.

The orbit space $M' = M/\xi$ of a KTS-space (M,g,ξ) admits a unique structure of a differentiable manifold such that the natural projection $\pi : M \to M'$ is a submersion. In fact, M' may be identified in a natural way with the coset space $G/K \cdot G^1$ and $G/K \to G/K \cdot G^1$ is a submersion. Note that ξ is a regular vector field.

The Lie group G^1 is isomorphic to either the circle group S^1 or to $\boldsymbol{R}$ depending on whether the leaves are compact or not. Here we identify S^1 with the set $\{e^{2\pi i t}, t \in \boldsymbol{R}\}$. If G^1 is a circle and l is the length of the integral curves of ξ, then S^1 acts freely on M on the right by

$$m \circ e^{2\pi i t} = \psi_{lt}(m) \tag{2.7}$$

for each $m \in M$. If G^1 is isomorphic to $\boldsymbol{R}$, we identify the action of $t \in \boldsymbol{R}$ on M with that of $\psi_t \in G^1$. In both cases M is a principal fibre bundle over M' with structural group G^1. The corresponding fundamental vector field ς generated by d/dt for $G^1 \approx S^1$ is given by

$$\varsigma(m) = \frac{d}{dt}_{|t=0} (m \circ e^{2\pi i t}) = l\xi_m \tag{2.8}$$

and in that case $l^{-1}\eta$ defines a connection form on M. For $G^1 \approx \boldsymbol{R}$, $\varsigma = \xi$ and then η is a connection form. Moreover, in the first case, using (2.1) and the fact that S^1 is Abelian, the usual structure equation is reduced to

$$\Omega = \frac{1}{l} d\eta = -\frac{1}{l} h$$

where Ω is the curvature form. Now, let h' be the (0,2)-tensor field on M' given by

$$h'(X',Y') = g'(H'X',Y')$$

for all $X',Y' \in \mathfrak{X}(M')$. Then $h = \pi^*(h')$ and the characteristic class $e_{M'}(M) \in H^2(M',\boldsymbol{Z})$ of M over M' (see [17]) satisfies

$$e_{M'}(M) = \left[-\frac{1}{l} h'\right]. \tag{2.9}$$

In what follows we shall consider the KTS-space (M,g,ξ) as a principal bundle over $M' = M/\xi$ with the description given above.

We have

THEOREM 2.4. *Let (M,g,ξ) be a* KTS-*space. Then the base space (M',g')*

is a symmetric space. The geodesic symmetry $s'_{m'}$ at $m'=\pi(m)$, $m\in M$, satisfies

$$s'_{m'}\circ\pi=\pi\circ s_m \tag{2.10}$$

PROOF. (2.10) is a consistent definition of $s'_{m'}$, $m'\in M'$ and we derive

$$(s'_{m'})_*(m')=-I$$

from it on $T_{m'}M'$. Further, since

$$s_{m*}X'^*=(s'_{m'*}X')^* \tag{2.11}$$

where $X'\in\mathfrak{X}(M')$, we see that $s'_{m'}$ is an isometry with respect to g'. So the required result follows. ■

We finish this section with a uniqueness result for KTS-spaces which will be used later.

PROPOSITION 2.7. *Let (M_1,g_1,ξ_1) and (M_2,g_2,ξ_2) be simply connected* KTS-*spaces over a symmetric space (M',g') which have the same field H' on M'. Then there exists an isometry $f:(M_1,g_1)\to(M_2,g_2)$ such that $f_*\xi_1\to\xi_2$.*

PROOF. Let $o_1\in M_1$ and $o_2\in M_2$ be such that $\pi_1(o_1)=\pi_2(o_2)$. since π_1 and π_2 are Riemannian submersions, the endomorphism $L:T_{o_1}M_1\to T_{o_2}M_2$ defined by

$$LX'_{o_1}{}^*=X'_{o_2}{}^*\quad,\quad L\xi_{1o_1}=\xi_{2o_2}$$

where $X'\in T_{\pi_1(o_1)}M'$, is an isometry and moreover

$$L\circ H_1=H_2\circ L. \tag{2.12}$$

Using the homogeneous structures T_1 and T_2 defined as in (2.5), we obtain

$$LT_{1U}V=T_{2LU}LV$$

and hence,

$$L\overline{K}_1(U,V)=\overline{K}_2(LU,LV)$$

where $U,V\in T_{o_1}M_1$ and $\overline{K}_1$ (respectively $\overline{K}_2$) is the torsion of the canonical connection $\overline{\nabla}_1$ (respectively $\overline{\nabla}_2$).

Further, (2.12) and (2.3) yield

$$LR_{1UV}W=R_{2LULV}LW$$

and hence, with (2.6):

$$L\overline{R}_{1UV}W=\overline{R}_{2LULV}LW.$$

Moreover, applying Proposition 2.2, we have $\overline{\nabla}_1\overline{K}_1=\overline{\nabla}_2\overline{K}_2=\overline{\nabla}_1\overline{R}_1=\overline{\nabla}_2\overline{R}_2=0$. So there exists a unique affine isomorphism f of M_1 onto M_2 such that $f_{*_{o_1}} = L$ [18]. In addition, since $\overline{\nabla}_1$ and $\overline{\nabla}_2$ are metric and L is an isometry, f is an isometry. Finally, since ξ_1 and ξ_2 are Killing vector fields, we obtain $f_*\xi_1 = \xi_2$ which completes the proof. ■

3. Contact KTS-spaces

Let (M, g) be a Riemannian manifold equipped with a contact flow $\mathfrak{F}_\xi$. From now on we will consider *contact* KTS-*spaces*.

First we have

PROPOSITION 3.1 [13]. *Let $\mathfrak{F}_\xi$ be a normal contact flow on a simply connected, complete Riemannian manifold (M,g). Then (M,g,ξ) is a (contact)* KTS-*space if and only if it is a naturally reductive homogeneous space with invariant unit vector field ξ.*

Further, using [31, Proposition 6.10], we get

PROPOSITION 3.2. *A contact* KTS-*space M is an irreducible Riemannian manifold and its homogeneous holonomy group coincides with the group $SO(n)$ of all isometries* $(n = \dim M)$.

Next, we consider the symmetric base space $(M' = M/\xi, g')$ of a contact KTS-space (M,g,ξ). Let $A(M')$ denote the group of all its H'-preserving isometries. Since it is a closed subgroup of $\mathfrak{I}(M')$, it is a Lie transformation group of M'. Since the reflections are H-preserving it follows, using (2.11), that H' is invariant under the geodesic symmetries of M'. Hence $A(M')$ contains all the symmetries and so, it acts transitively on M'. Moreover, $\mathfrak{I}_o(M')$ is semi-simple if and only if $A_o(M')$ is semi-simple and in this case $A_o(M') = \mathfrak{I}_o(M')$ (see [14, proof of Lemma 4.3, Chapter VIII]). For a more detailed study we start with the case of an irreducible M'. We have

THEOREM 3.1. *Let (M,g,ξ) be a contact* KTS-*space such that the base space (M',g') is an irreducible symmetric space. Then the sectional curvature $K(X,\xi)$, X horizontal, is a non-vanishing constant $k = c^2 > 0$. Moreover, $(M, c^2 g, c^{-1}H, c^{-1}\xi, c\eta)$ is a φ-symmetric space over the Hermitian symmetric space $(M', c^2 g', c^{-1}H')$.*

PROOF. M' may be represented as G/K where $G = \mathfrak{I}_o(M')$ is semi-simple and acts effectively on M', and K is compact [14, Theorem 4.1, Chapter V]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the canonical decomposition of the Lie algebra $\mathfrak{g}$ of G. Under the identification of $\mathfrak{m}$ with $T_{o'}M'$, o' being the coset K, $H'_{o'}$ defines a linear endomorphism of $\mathfrak{m}$ which we extend to $\mathfrak{g}$ by $H'_{o'}X = 0$ for $X \in \mathfrak{k}$. We shall prove that $H'_{o'}$ is a derivation of $\mathfrak{g}$, that is,

$$H'_{o'}[X,Y] = [H'_{o'}X,Y] + [X,H'_{o'}Y],$$

$X,Y \in \mathfrak{g}$. First, this identity is trivial for $X,Y \in \mathfrak{k}$. For $X \in \mathfrak{k}$ and $Y \in \mathfrak{m}$ it is satisfied since H' is $\mathrm{Ad}(K)$-invariant. Finally, let $X,Y \in \mathfrak{m}$. Since $R'_{XY} = -ad_{\mathfrak{m}}([X,Y])$, (2.4) yields

$$ad_{\mathfrak{m}}[H'_{o'}X,Y] + ad_{\mathfrak{m}}[X,H'_{o'}Y] = 0.$$

Now, using the fact that the representation $\mathfrak{k} \to ad_{\mathfrak{m}}(\mathfrak{k})$ is faithful, we get from this:

$$[H'_{o'}X,Y] + [X,H'_{o'}Y] = 0$$

and so, $H'_{o'}$ is a derivation.

Now, since G is semi-simple, every derivation of $\mathfrak{g}$ is inner [5, Corollaire 3, p. 73]. So there is an element $Z_o \in \mathfrak{g}$ such that $H'_{o'} = ad(Z_o)$. As $H'_{o'}$ is of maximal rank $n-1 = \dim M'$, $Z_o \in \mathfrak{k}$ and consequently, Z_o is in the center of $\mathfrak{k}$. Now, using [18, Proposition 7.5, Chapter XI] we get that $\mathfrak{g}$ is simple and applying [18, Theorem 9.6, Chapter XI] M' admits an invariant Hermitian structure. From [14, Proposition 6.2, Chapter VIII] the center $Z(K)$ of K is one-dimensional and analytically isomorphic to the circle group. The complex structure $J_{o'}$ on $\mathfrak{m}$ can be expressed as $J_{o'} = ad_{\mathfrak{m}}(Z_1)$ where Z_1 belongs to the center of $\mathfrak{k}$. Then, there exists a non-vanishing constant c such that $Z_o = cZ_1$ and so $H'_{o'} = cJ_{o'}$. Thus, $(M',g',c^{-1}H')$ is a Hermitian symmetric space. Then, using Lemma 2.1, we get $K(X,\xi) = c^2$ for all horizontal X. The rest follows now by direct computation or as in [8, Theorem 3.2] and so $(M',c^2g,c^{-1}H,c^{-1}\xi,c\eta)$ is a φ-symmetric space. ■

REMARK 3.1. Note that if in the situation of Theorem 3.1, M fibers over M' such that we have a principal S^1-bundle and if Φ denotes the Kähler form of (M',g',J) (that is, $\Phi(X',Y') = g'(JX',Y')$, $X',Y' \in \mathfrak{X}(M')$), then the characteristic class $e_{M'}(M)$ coincides with the cohomology class $[-\frac{c}{l}\Phi]$ where l is the length of the fibers.

Further we prove

THEOREM 3.2. *The base space* (M',g') *of a simply connected contact* KTS-*space* (M,g,ξ) *is a (simply connected) Hermitian symmetric space. Moreover, if* $M'=M_0'\times M_1'\times\cdots\times M_r'$ *is its de Rham decomposition and* $\mathcal{H}_i$, $i=0,\cdots,r$, *are the smooth distributions on M obtained by the horizontal lifts of the tangent vectors to* M_i', *then, for each* $m\in M$, $\mathcal{H}(m)=\mathcal{H}_0(m)\oplus\mathcal{H}_1(m)\oplus\cdots\oplus\mathcal{H}_r(m)$ *is an* H-*invariant orthogonal decomposition of the horizontal subspace* $\mathcal{H}(m)$ *and each sectional curvature* $K(\mathcal{H}_j,\xi)$, $j=1,\cdots,r$, *is a positive constant.*

PROOF. Let (M,g,ξ) be a simply connected contact KTS-space over a symmetric space M'. From the existence of local sections of the submersion $\pi:M\to M'$ and from the fact that the fibers are connected one gets that M' is also simply connected. Let $G=A_o(M')$ and let K denote the isotropy subgroup of G at some point o' of M'. If $(\tilde G,\Psi)$ is the universal covering group of G and $\tilde K$ the identity component of $\Psi^{-1}(K)$, then $M'=\tilde G/\tilde K$.

Next, let $M'=M_0'\times M_1'\times\cdots\times M_r'$ be the de Rham decomposition of M' where M_0' is a Euclidean space and $M_1',\cdots,M_r'$ are irreducible symmetric spaces. Further, $\tilde G$ and $\tilde K$ admit decompositions $\tilde G=G_0\times G_1\times\cdots\times G_r$ and $\tilde K=K_0\times K_1\times\cdots\times K_r$ such that $M_i'=G_i/K_i$, $i=0,1,\cdots,r$. Consider also the compact and non-compact factors M_+' and M_-' of M'. Then $M'=M_0'\times M_+'\times M_-'$, $\tilde G=G_0\times G_+\times G_-$, $\tilde K=K_0\times K_+\times K_-$ and $M_0'=G_0/K_0, M_+'=G_+/K_+, M_-'=G_-/K_-$. Let $\mathfrak{m}$, $\mathfrak{m}_0$, $\mathfrak{m}_+$, $\mathfrak{m}_-$ denote the eigenspaces for the eigenvalue -1 of σ_* where σ is the automorphism of G given by $\sigma:g\mapsto s_{o'}'gs_{o'}'$. As usual these eigenspaces may be identified with tangent spaces to M', M_0', M_+', M_-' and $\mathfrak{m}=\mathfrak{m}_0\oplus\mathfrak{m}_+\oplus\mathfrak{m}_-$.

Since $H_{o'}':\mathfrak{m}\to\mathfrak{m}$ is Ad(K)-invariant, it follows that $\mathfrak{m}_0$, $\mathfrak{m}_+$, $\mathfrak{m}_-$ are invariant under $H_{o'}'$ (see proof of Proposition 4.4 in Chapter VIII of [14]). As G_+ and G_- are semi-simple Lie groups and $H_{o'}'$ is of maximal rank, it follows as in the proof of Theorem 3.1, that there is a $Z_+\in Z(\mathfrak{k}_+)$ and a $Z_-\in Z(\mathfrak{k}_-)$ such that $H_{o'|\mathfrak{m}_+}'=ad_{\mathfrak{m}_+}(Z_+)$ and $H_{o'|\mathfrak{m}_-}'=ad_{\mathfrak{m}_-}(Z_-)$. But, $Z_++Z_-=Z_1+\cdots+Z_r$ where $Z_j\in Z(\mathfrak{k}_j)$, $j=1,\cdots,r$, and this yields

$$H_{o'}'=H_{o'|\mathfrak{m}_0}'\times ad_{\mathfrak{m}_1}(Z_1)\times\cdots\times ad_{\mathfrak{m}_r}(Z_r).$$

So, each $M_j'=G_j/K_j$ admits an invariant Hermitian structure J_j and there exist non-vanishing constants $c_1,\cdots,c_r$ such that

$$ad_{\mathfrak{m}_j}(Z_j)=c_jJ_j. \tag{3.1}$$

Then the linear operator on $\mathfrak{m}_+\oplus\mathfrak{m}_-$ given by

$$\frac{1}{c_1}ad_{\mathfrak{m}_1}(Z_1)\times\cdots\times\frac{1}{c_r}ad_{\mathfrak{m}_r}(Z_r)$$

defines an almost complex structure on $M'_+ \times M'_-$. Hence, taking also into account that the dimension of M'_0 is even, we have a Hermitian structure on M'. Lemma 2.1 yields that the sectional curvatures $K(\mathcal{H}_j, \xi) = c_j^2$, $j = 1, \cdots, r$. ■

4. Construction of contact KTS-spaces

In this section we will construct several classes of contact KTS-spaces which will play a fundamental role in the classification.

4.1. Contact KTS-spaces over irreducible Hermitian symmetric spaces

Let $M' = G/K$ be an n'-dimensional irreducible Hermitian symmetric space where G is a connected simple Lie group with trivial center and acting effectively on M', and K connected and compact. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}^-$ be the canonical decomposition of the Lie algebra $\mathfrak{g}$. Since all G-invariant Riemannian metrics on M' coincide up to a constant factor, we assume that, under the identification of $\mathfrak{m}^-$ with $T_{o'}M'$, o' being the coset K, the Riemannian metric on M' is given by $\beta B_{\mathfrak{m}^-}$ where B is the Killing form of $\mathfrak{g}$. $\beta < 0$ if G is compact and $\beta > 0$ if G is non-compact. The center of $\mathfrak{k}$ is one-dimensional and there exists a $Z_o \in Z(\mathfrak{k})$ such that $J_o = ad_{\mathfrak{m}^-}(Z_o)$ defines the corresponding complex structure J on M'. Since $\mathfrak{k}$ is compact, we have $\mathfrak{k} = \mathfrak{h} \oplus \boldsymbol{R}Z_o$ where $\mathfrak{h}$ is the compact semi-simple subalgebra $[\mathfrak{k}, \mathfrak{k}]$. Let $H \subset K$ be the connected subgroup associated to $\mathfrak{h}$. Then H is also compact and $M = G/H$ is a manifold with $\mathfrak{m} = \mathfrak{m}^- \oplus \boldsymbol{R}Z_o$ as tangent space at the origin. Moreover, the canonical projection $\pi : M \to M'$ defines a principal S^1-bundle.

Using [15] (see also [16], [25]), $M = G/H$ is a globally φ-symmetric space and an invariant Sasakian structure on M is determined by the $\mathrm{Ad}(H)$-invariant tensors $(g_o, \varphi_o, \xi_o, \eta_o)$ on $\mathfrak{m}$ where g_o is the inner product βB *on* $\mathfrak{m}^-$, $g_o(\xi_o, \xi_o) = 1$ and by supposing that the decomposition $\mathfrak{m}^- \oplus \boldsymbol{R}Z_o$ is orthogonal, $\varphi_o = ad_{\mathfrak{m}}(Z_o)$, $\xi_o = -(1/2n'\beta)Z_o$ and where η_o is the one-form on $\mathfrak{m}$ such that $\eta_o(\xi_o) = 1$, $\eta_o(\mathfrak{m}^-) = 0$. Similarly, one can define, for each $c \in \boldsymbol{R} - \{0\}$, an invariant Sasakian structure determined by the $\mathrm{Ad}(H)$-invariant tensors $\varphi_o, c^{-2}\xi_o$ and the inner product $\langle , \rangle$ defined by $c^2\beta B$ on $\mathfrak{m}^-$, $\langle c^{-2}\xi_o, c^{-2}\xi_o \rangle = 1$ and by supposing that $\mathfrak{m}^- \oplus \boldsymbol{R}, Z_o$ is on orthogomal, decomposition. With this structure M fibers again over M' with Riemannian metric given by $c^2\beta B_{\mathfrak{m}^-}$. Now, from Theorem 3.1, we can construct KTS-structures on M which makes from π a Riemannian submersion. More specifically, these structures are given by

$$\left(\xi_c = -\frac{1}{2n'c\beta} Z_o, g_c\right) \tag{4.1}$$

where the inner product g_c is defined as g_o with $g_c(\xi_c,\xi_c)=1$. Here, $H'_c=cJ_o$ and the sectional curvature $K(X,\xi)=c^2$ for all horizontal X.

Let $\tilde{H}$ and $\tilde{K}$ be the connected subgroups of $\tilde{G}$, the universal covering of G, associated to the Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, respectively. Then the universal covering $\tilde{M}$ of M may be written as $\tilde{G}/\tilde{H}$ and M' as $\tilde{G}/\tilde{K}$. In this case $\tilde{M}$ is a principal fiber bundle over M' via the above construction with group $\tilde{K}/\tilde{H}$. Note that $\tilde{H}$ is also compact since $\mathfrak{h}$ is compact and semi-simple. On $\tilde{M}$ we get the corresponding KTS-structures $(\tilde{g}_c,\tilde{\xi}_c)$ (see Corollary 2.1).

So, using Proposition 2.7 and Theorem 3.1 we obtain

THEOREM 4.1. *Let $(M'=G/K, g'=\beta B, J)$ be an n'-dimensional irreducible Hermitian symmetric space. Then each simply connected* KTS*-space over (M',g') is isomorphic to $(\tilde{M}=\tilde{G}/\tilde{H},\tilde{g}_c,\tilde{\xi}_c)$ for some $c\in \mathbf{R}-\{0\}$.*

When G is compact, then $\tilde{G}$ is also compact [14, Theorem 6.9, p. 133]) and so, $\tilde{\pi}:\tilde{M}\to M'$ is a principal S^1-bundle. The length of the fibers on $(\tilde{M},\tilde{g}_c,\tilde{\xi}_c)$ depends on c. In fact, if l_c and $l_{c'}$ are the lengths of the integral curves of $\tilde{\xi}_c$ and $\tilde{\xi}_{c'}$ in $(\tilde{M},\tilde{g}_c)$ and $(\tilde{M},\tilde{g}_{c'})$, respectively, then we have

$$\frac{l_c}{l_{c'}}=\left|\frac{c}{c'}\right|. \tag{4.2}$$

(2.8) yields that the fundamental vector fields generated by d/dt and its connection forms coincide up to sign for all non-zero c. Note that $(\tilde{M},\tilde{g}_{-c},\tilde{\xi}_{-c})$ corresponds, in the additive group of all principal S^1-bundles over M', with the inverse element of $(\tilde{M},\tilde{g}_c,\tilde{\xi}_c)$.

REMARK 4.1. To work with the characteristic class $e_{M'}(\tilde{M})$ we shall fix the scalar $\beta^o<0$ such that the corresponding Kähler form Φ^o satisfies $e_{M'}(\tilde{M})=[-(1/2\pi)\Phi^o]$. From Remark 3.1 it then follows that the length of the fibers of the simply connected φ-symmetric space $(\tilde{M},\tilde{g}^o,\tilde{\varphi}^o,\tilde{\xi}^o,\tilde{\eta}^o)$ over $(M',g'=\beta^o B)$ is precisely 2π. Moreover, using (4.2), the length l_c of the fibers of the KTS-space $(\tilde{M},\tilde{g}_c,\tilde{\xi}_c)$ satisfies

$$l_c=\left|\frac{2\pi c\beta}{\beta^o}\right|.$$

Now, let G_n be the cyclic subgroup of S^1 of order n. Then the quotient space $\tilde{M}/G_n$ is a principal S^1-bundle over M' (see [17, p. 37]) and it is a KTS-space with the natural structure that $(\tilde{g}_c,\tilde{\xi}_c)$ induces on $\tilde{M}/G_n$. We will use the same notation for this induced structure. It follows from [17, Theorem 11] that all principal fiber bundles over M' with structural group S^1 are of the form $\tilde{M}/G_n$, $n \in \boldsymbol{Z}$. Using the isomorphism between the additive group of all principal S^1-bundles over M' and the second cohomology group $H^2(M',\boldsymbol{Z})$ of M' with integer coefficients, it follows that $e_{M'}(\tilde{M}/G_n) = ne_{M'}(\tilde{M})$. From Remark 3.1 it follows that the length of the integral curves of $\tilde{\xi}_c$ on $(\tilde{M}/G_n, \tilde{g}_c)$ is l_c/n where l_c denotes the corresponding length on $(\tilde{M}, \tilde{g}_c)$. The fundamental vector field ς_n on $\tilde{M}/G_n$ satisfies $\varsigma_n = (1/n)\varsigma$ where ς is that on $\tilde{M}$.

Finally, we consider the case of a Hermitian symmetric space M' of non-compact type. Then $\tilde{M}$ fibers over M' defining a principal $\boldsymbol{R}^1$-bundle (see [15, Section II.5.2]). Now, let $\boldsymbol{Z}[l]$ denote the subgroup of $\boldsymbol{R}$ generated by an element $l \in \boldsymbol{R}$, $l \geq 0$. If $l > 0$, then $\tilde{M}/\boldsymbol{Z}[l]$ is a principal S^1-bundle over M' and $(\tilde{g}_c, \tilde{\xi}_c)$, $c \in \boldsymbol{R} - \{0\}$, again defines on $\tilde{M}/\boldsymbol{Z}[l]$ a class of KTS-structures. In this case, the length of the fibers on $(\tilde{M}/\boldsymbol{Z}[l], \tilde{g}_c, \tilde{\xi}_c)$ is independent of c, more precisely it is equal to l. However, from (2.8), (2.9) and Remark 3.1 we find that the fundamental vector field ς and the characteristic class $e_{M'}(\tilde{M}/\boldsymbol{Z}[l])$ are given by $\varsigma = l\tilde{\xi}_c$ and $e_{M'}(\tilde{M}/\boldsymbol{Z}[l]) = [-(c/l)\Phi]$.

4.2. Contact KTS-spaces over Hermitian symmetric spaces of compact type

Let $M'_+ = G/K$ be a Hermitian symmetric space of compact type and $M'_+ = M'_1 \times \cdots \times M'_r$ its de Rham decomposition into irreducible factors. Then $M'_i = G_i/K_i$, $i = 1, \cdots, r$, where each G_i is a connected, compact, simple Lie group with trivial center. The Riemannian metric on M'_+ is of the form $\langle , \rangle = \beta_1 B_1 \perp \cdots \perp \beta_r B_r$ where B_i is the Killing form of G_i and $\beta_i < 0$. From Theorem 4.1 we get that the simply connected contact KTS-spaces over each M'_i are given by $\tilde{M}_i = \tilde{G}_i/\tilde{H}_i, \tilde{g}_{c_i}, \tilde{\xi}_{c_i})$, $c_i \in \boldsymbol{R} - \{0\}$, where $\tilde{g}_{c_i}$ denotes the unique invariant metric determined by ($\beta_i B_i$ and ξ_{c_i}) following the construction given in Section 4.1. We identify these principal S^1-bundles with the product manifolds $M'_1 \times \cdots \times \tilde{M}_i \times \cdots \times M'_r$ as principal S^1-bundles over M'_+. From [17, Theorem 11], since $M'_+ = \tilde{G}/\tilde{K}$ and $\tilde{K}$ is compact, $\{\tilde{M}_i, \ i = 1, \cdots, r\}$ generate the group of all principal S^1-bundles over M'_+ or equivalently, the set of their characteristic classes $\{e_{M'_i}(\tilde{M}_i), \ i = 1, \cdots, r\}$ generates $H^2(M'_+, \boldsymbol{Z})$.

Let (M_+, g, ξ) be a KTS-space which fibers over M'_+. If M_+ is a principal $\boldsymbol{R}^1$-bundle, then, for each $l > 0$, $M_+/\boldsymbol{Z}[l]$ is a principal S^1-bundle and so, we may suppose that (up to a covering) M_+ is a principal circle bundle over M'_+. Let l denote the length of its fibers. Then there exist integers $n_1, \cdots, n_r$ such that

(4.3) $$e_{M'_+}(M_+) = \sum_{i=1}^{r} n_i e_{M'_i}(\tilde{M}_i).$$

Now, from Remark 4.1 we have

$$e_{M'_i}(\tilde{M}_i) = \left[-\frac{1}{l_{c_i}} p_i^* h'_{c_i} \right] = \left[-\frac{1}{2\pi} p_i^* \Phi_i^o \right],$$

$i = 1, \cdots, r$, where p_i denotes the projection of M'_+ onto M'_i. So, using Theorem 3.2, we get

$$e_{M'_+}(M_+) = \left[-\frac{1}{l} h' \right] = \left[-\frac{1}{l} \sum_{i=1}^{r} c_i p_i^* \Phi_i \right] = \frac{1}{l} \sum_{i=1}^{r} \frac{2\pi c_i \beta_i}{\beta_i^o} e_{M'_i}(\tilde{M}_i)$$

and hence, by comparing this with (4.3), we obtain

(4.4) $$c_i = \frac{l n_i \beta_i^o}{2\pi \beta_i}.$$

Taking into account Remark 4.1 this may also be written as

(4.5) $$c_i = \frac{l n_i}{l_i},$$

where l_i denotes the length of the fibers of the φ-symmetric space $(\tilde{M}_i, \tilde{g}_i, \tilde{\varphi}_i, \tilde{\xi}_i, \tilde{\eta}_i)$ over $(M'_i, \beta_i B_i)$. From Lemma 2.1 and Theorem 3.2 we get that $K(\mathcal{H}_i, \xi) = c_i^2$ for each $i \in \{1, \cdots, r\}$ and so

$$K(\mathcal{H}_i, \xi) = \left(\frac{l n_i}{l_i} \right)^2.$$

Moreover, the (1, 1)-tensor H' *on* M'_+ satisfies

(4.6) $$H' = \frac{l n_1}{l_1} J_1 \times \cdots \times \frac{l n_r}{l_r} J_r.$$

Now, suppose that M_+ is a principal circle bundle over M'_+ with characteristic class $e_{M'_+}(M_+) = \sum_{i=1}^{r} n_i e_{M'_i}(\tilde{M}_i)$. Then, for each $l > 0$, M_+ carries a KTS-structure (g_l, ξ_l) over M'_+ where the length of its integral curves is precisely l. In fact, from [17, Theorem 5 and Theorem 11] M_+ may be written as

$$M_+ = (\tilde{M}_1 / G_{n_1} \times \cdots \times \tilde{M}_r / G_{n_r}) / T^{r-1}$$

where T^{r-1} denotes the $(r-1)$-dimensional torus and the action of T^{r-1} on $\tilde{M}_1 / G_{n_1} \times \cdots \times \tilde{M}_r / G_{n_r}$ is defined by

(4.7) $$(m_1, \cdots, m_r)(s_2, \cdots, s_r) = \left(m_1 \prod_{i=2}^{r} s_i, m_2 s_2^{-1}, \cdots, m_r s_r^{-1} \right).$$

Let $\rho:\tilde{M}_1/G_{n_1}\times\cdots\times\tilde{M}_r/G_{n_r}\to M_+$ be the canonical projection and put $[(m_1,\cdots,m_r)]=\rho(m_1,\cdots,m_r)$. Then the action of S^1 on M_+ is given by

$$[(m_1,\cdots,m_r)]s=[(m_1s,\cdots,m_r)]=\cdots=[(m_1,\cdots,m_rs)]$$

and the projection $\pi:M_+\to M'_+$ is defined by $\pi\circ\rho=\pi_1\times\cdots\times\pi_r$ where π_i $i=1,\cdots,r$ is the projection of $\tilde{M}_i/G_{n_i}$ onto M'_i. So, the fundamental vector field ς generated by d/dt on M_+ is determined by $\varsigma=\rho_*\varsigma_i$ for any $i\in\{1,\cdots,r\}$, where ς_i denotes the fundamental vector field on $\tilde{M}_i/G_{n_i}$. From (2.8) we have $\varsigma_i=(l_i/n_i)\tilde{\xi}_i$ and so,

$$\varsigma=\frac{l_i}{n_i}\rho_*\tilde{\xi}_i.$$

Now, put $\xi_l=(1/l)\varsigma$, l being an arbitrary positive real number. Then, by choosing c_i as in (4.5) for $1\le i\le r$, we get

$$\xi_l=\frac{i}{c_i}\rho_*\tilde{\xi}_i. \tag{4.8}$$

Each one-form $(n_i/l_i)\tilde{\eta}_i$ is a connection form on M_i/G_{n_i} and hence, $(1/l)\eta_l$, where η_l is the unique differential form on M_+ such that

$$\rho^*\eta_l=c_1\tilde{\eta}_1\times\cdots\times c_r\tilde{\eta}_r,$$

defines a connection form on M_+ (see [17]). (4.8) yields $\eta_l(\xi_l)=1$. Moreover, for $X'\in\mathfrak{X}(M'_+)$ we have

$$X'^*=\rho_*((p_{1*}X')^*,\cdots,(p_{r*}X')^*).$$

Now, we denote by g_l the unique Riemannian metric on M_+ such that $g_l(\xi_l,\xi_l)=1$, ξ_l is orthogonal to $\ker\eta_l$ and $\pi:M_+\to M'_+$ becomes a Riemannian submersion. Then ξ_l is a unit Killing vector field on (M_+,g_l) and the length of its integral curves is precisely l.

For each $m=[(m_1,\cdots,m_r)]\in M_+$ there is a unique function $s_m:M_+\to M_+$ satisfying

$$s_m\circ\rho=\rho\circ(s^1_{m_1}\times\cdots\times s^r_{m_r})$$

where $s^i_{m_i}$ denotes the reflection of $\tilde{M}_i/G_{n_i}$ at m_i. It then follows easily that s_m is the reflection at m and hence (M_+,g_l,ξ_l) is a contact KTS-space.

Thus, using Proposition 2.7 and (4.4), (4.6) we may conclude

THEOREM 4.2. *Let M'_+ be a Hermitian symmetric space of compact type and $M'_+=M'_1\times\cdots\times M'_r$ its de Rham decomposition. Then each simply connected contact* KTS-*space over M'_+ is isomorphic, for some $l>0$ and non-vanishing* *integers* $n_1,\cdots,n_r$, *to* $(\tilde{M}_+,\tilde{g}_l,\tilde{\xi}_l)$ *where*

$$M_+=(\tilde{M}_i/G_{n_i}\times\cdots\times\tilde{M}_r/G_{n_r})/T^{r-1}.$$

The sectional curvature $K(\mathcal{H}_i,\tilde{\xi}_l)$ *on* $\tilde{M}_+$, $i=1,\cdots,r$ (*where* $\mathcal{H}_i$ *is defined as in* Theorem 3.2) *is constant and satisfies*

$$K(\mathcal{H}_i,\tilde{\xi}_l)=\left(\frac{ln_i\beta_i^o}{2\pi\beta_i}\right)^2,$$

where β_i^o *is the negative scalar such that the corresponding Kähler form* Φ_i^o *on* M_i' *verifies* $e_{M_i'}(\tilde{M}_i)=\left[-\frac{1}{2\pi}\Phi_i^o\right]$.

REMARK 4.2. Theorem 4.2 gives the existence of contact KTS-spaces over an arbitrary Hermitian symmetric space of compact type. However, for φ-symmetric spaces such existence is subjected to certain restrictions on the metrics of the base spaces (see [15, Theorem A]).

Let $M_+'=G/K$ be an irreducible Hermitian symmetric space of compact type. On M_+' we consider the Riemannian metrics g_1 and g_2 determined by $\beta_1 B$ and $\beta_2 B$, respectively, where B is the Killing form of the Lie algebra $\mathfrak{g}$ of G. If β_1/β_2 is an irrational number, then it follows from [15, Proposition III.2] or directly from (4.4) that there do not exist any φ-symmetric space which fibers over the product symmetric space $(M_+',g_1)\times(M_+',g_2)$.

We also have that for $\beta_1\neq\beta_2$ the quotient space

$$(\tilde{M}_+\times\tilde{M}_+)/S^1$$

is a contact KTS-space but it does not admit a structure of a φ-symmetric space over $M_+'\times M_+'$ (see [15]).

4.3. Contact KTS-spaces over Hermitian symmetric spaces of non-compact and of Euclidean type

Let M_-' be a Hermitian symmetric space of non-compact type and $M_-'=M_1'\times\cdots\times M_r'$ its de Rham decomposition. In a similar way as in Section 4.2 we get the coset space

$$M_-'=(M_1'\times\cdots\times M_r')/\boldsymbol{R}^{r-1}$$

as a simply connected principal $\boldsymbol{R}^1$-bundle over M_-' where the action of $\boldsymbol{R}^{r-1}$ on $\tilde{M}_1\times\cdots\times\tilde{M}_r$ is defined by

$$(m_1,\cdots,m_r)(t_2,\cdots,t_r)=\left(m_1\sum_{i=2}^{r}t_i,m_2(-t_2),\cdots,m_r(-t_r)\right).$$

Note that this action, just as that given by (4.7), depends on the KTS-structure $(\tilde{g}_{c_i},\tilde{\xi}_{c_i})$ on $\tilde{M}_i$. The fundamental vector field ς generated by d/dt on M_- is determined by

$$\varsigma=\rho_*\tilde{\xi}_{c_i}$$

for any $i\in\{1,\cdots,r\}$ where ρ denotes the canonical projection $\rho:\tilde{M}_1\times\cdots\times\tilde{M}_r\to M_-$. We put $\xi=\varsigma$. Again, we define the connection form η on M_- as the unique differential form on M_- such that

$$\rho^*\eta=\tilde{\eta}_{c_1}\times\cdots\times\tilde{\eta}_{c_r}.$$

If g denotes the Riemannian metric on M_- constructed as in Section 4.2, we get that (M_-,g,ξ) is a KTS-space. From (2.1) it follows that the corresponding (0,2)-tensor h is determined by

$$\rho^*h=\tilde{h}_{c_1}\times\cdots\times\tilde{h}_{c_r}$$

and consequently, for the (1,1)-tensor H' on M'_- we get

$$H'=c_1J_1\times\cdots\times c_rJ_r. \tag{4.9}$$

Thus, from Proposition 2.7 and Theorem 3.2 we have

THEOREM 4.3. *Let M'_- be a Hermitian symmetric space of non-compact type with the de Rham decomposition $M'_-=\tilde{M}_1\times\cdots\times\tilde{M}_r$. Then each simply connected contact* KTS-*space over M'_- is isomorphic to*

$$(M_-=(\tilde{M}_1\times\cdots\times\tilde{M}_r)/\boldsymbol{R}^{r-1},g,\xi)$$

where the KTS-*structures (g,ξ) defined as above depend on the r real parameters $c_1,\cdots,c_r$.*

From (4.9) it follows that for each $l>0$ the characteristic class of the principal S^1-bundle $M_-/Z[l]$ over M'_- satisfies

$$e_{M'_-}(M_-/Z[l])=\sum_{i=2}^{r}e_{M'_-}(\tilde{M}_i/Z[l]).$$

Next, let $H(p,1)$, $p\geq 1$, be the connected, simply connected nilpotent Lie group of (real) matrices of the form

$$A=\begin{pmatrix} 1 & a_1^2 & \cdots & a_1^{p+1} & a_1^{p+2} \\ 0 & 1 & \cdots & 0 & a_2^{p+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{p+1}^{p+2} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

that is, the $(2p+1)$-dimensional Heisenberg group. Denote by y_k, y_{p+k}, z, $1\le k\le p$, the coordinate functions on $H(p,1)$ defined by

$$y_k(A)=a_1^{k+1} \quad , \quad y_{p+k}(A)=a_{k+1}^{p+2} \quad , \quad z(A)=a_1^{p+2}$$

for any $A\in H(p,1)$. Then we have

THEOREM 4.4. *The simply connected contact* KTS-*spaces over the Euclidean space* E^{2p} *are isomorphic to* $(H(p,1), g_{\lambda_1\cdots\lambda_p}, \ \xi=\partial/\partial z)$ *where*

$$g_{\lambda_1\cdots\lambda_p}=\sum_{k=1}^{p}\left\{\frac{1}{\lambda_k}\left((dy_k)^2+(dy_{p+k})^2\right)\right\}+\left(dz-\sum_{1\le l\le p}y_l dy_{p+l}\right)^2 \tag{4.10}$$

and $\lambda_1,\cdots,\lambda_p$ *are positive real parameters.*

PROOF. It follows from Proposition 2.7 that the different simply connected KTS-spaces over a given symmetric space M' are related to the possible parallel skew-symmetric tensor fields H' on M' where as mentioned before $H'X'=\pi_*(HX'^*)$. On E^{2p} we can choose a coordinate system $(x_1,\cdots,x_{2p})$ such that H' is defined by

$$H'\frac{\partial}{\partial x_k}=\mu_k\frac{\partial}{\partial x_{p+k}}, \quad H'\frac{\partial}{\partial x_{p+k}}=-\mu_k\frac{\partial}{\partial x_k} \tag{4.11}$$

where $\mu_1,\cdots,\mu_p$ are positive real numbers.

Then, using [8], it follows that $(H(p,1), g_{\lambda_1\cdots\lambda_p}, \xi=\partial/\partial z)$ is a KTS-space. With the projection $\pi: H(p,1)\to E^{2p}$ given by

$$(y_1,\cdots,y_{2p},z)\mapsto\left(\frac{1}{\sqrt{\lambda_1}}y_1,\cdots,\frac{1}{\sqrt{\lambda_p}}y_p,\frac{1}{\sqrt{\lambda_1}}y_{p+1},\cdots,\frac{1}{\sqrt{\lambda_p}}y_{2p}\right)$$

$H(p,1)$ is a principal $\boldsymbol{R}^1$-bundle which is a Riemannian submersion and the corresponding tensor field H' on E^{2p} coincides with that given in (4.11) putting $\mu_k=\lambda_k/2$, $k=1,\cdots,p$. ■

The characteristic class of the principal S^1-bundle $H(p,1)/\boldsymbol{Z}[l]$ over E^{2p} is given, in terms of the coordinate system $(x_1,\cdots,x_{2p})$ on E^{2p} by

$$e_{E^{2p}}(H(p,1)/\boldsymbol{Z}[l])=\left[-\frac{1}{l}\sum_{k=1}^{p}\lambda_k dx_k\wedge dx_{p+k}\right].$$

Finally, consider the Hermitian symmetric space $M'=E^{2p}\times M'_-$. Since $H(p,1)$ and M_- are principal $\boldsymbol{R}^1$-bundles over E^{2p} and M'_-, respectively, the manifold

$$N=(H(p,1)\times M_-)/\boldsymbol{R}$$

is a simply connected principal $\boldsymbol{R}^1$-bundle over M' as in the non-compact case. Following the notations as above, the (0,2)-tensor field h' on M' is given by

$$h'=\left(\sum_{k=1}^{p}\lambda_k dx_k\wedge dx_{p+k}\right)\times c_1\Phi_1\times\cdots\times c_r\Phi_r.$$

Hence, using again Proposition 2.7 we obtain that each simply connected contact KTS-space over M' is isomorphic to (N,g,ξ) where the family of KTS-structures (g,ξ) now depends on the $p + r$ parameters $\lambda_1,\cdots,\lambda_p,c_1,\cdots,c_r$.

4.4 Contact KTS-spaces over a Hermitian symmetric space

Let $M'=E^{2p}\times M'_-\times M'_+$ be a simply connected Hermitian symmetric space where E^{2p}, M'_- and M'_+ are, respectively, of Euclidean, non-compact and compact type. Let $M'_-=M'_1\times\cdots\times M'_q$ and $M'_+=M'_{q+1}\times\cdots\times M'_r$ be the decompositions into irreducible Hermitian symmetric spaces of M'_- and M'_+. Consider also the contact KTS-space (M_+,g_l,ξ_l) over M'_+ where M_+ is the following quotient manifold (see Theorem 4.2):

$$M_+=(\tilde{M}_{q+1}/G_{n_{q+1}}\times\cdots\times\tilde{M}_r/G_{n_r})/T^{r-q-1}$$

and where $l > 0$ is the length of the fibers. The coset space $N/\boldsymbol{Z}[l]$ where $N=(H(p,1)\times M_-)/\boldsymbol{R}$ is also a principal S^1-bundle with the same length l for its fibers. Then for each $l > 0$ we construct non-isomorphic contact KTS-structures on the manifold

$$M=(N/\boldsymbol{Z}[l]\times M_+)/S^1$$

considered as principal S^1-bundles over $M'=E^{2p}\times M'_-\times M'_+$.

Using the method given in Section 4.2, we may obtain KTS-structure (g,ξ) such that the corresponding tensor field h' of type (0,2) is given by

$$h'=\left(\sum_{k=1}^{p}\lambda_k dx_k\wedge dx_{p+k}\right)\times\prod_{i=1}^{q}c_i\Phi_i\times\prod_{j=1}^{r-q}c_{q+j}\Phi_{q+j}$$

with

$$c_a = \frac{ln_a\beta_a^o}{2\pi\beta_a}, \qquad a = q+1,\cdots,r.$$

So we obtain, by Proposition 2.7,

THEOREM 4.5. *Let $M' = E^{2p} \times M'_- \times M'_+$ be a simply connected Hermitian symmetric space and $M'_- = M'_1 \times \cdots \times M'_q$ and $M'_+ = M'_{q+1} \times \cdots \times M'_r$ the de Rham decompositions of M'_- and M'_+ respectively. Then each simply connected contact* KTS*-space over M' is isomorphic, for some $l > 0$ and some non-vanishing integers $n_{q+1},\cdots,n_r$, to $(\tilde{M},\tilde{g},\tilde{\xi})$ where*

$$M = (N/Z[l] \times M_+)/S^1,$$

$N = (H(p,1) \times M_-)/\boldsymbol{R}$ and $M_+ = (\tilde{M}_{q+1}/G_{n_{q+1}} \times \cdots \times \tilde{M}_r/G_{n_r})/T^{r-q-1}$. The family of KTS*-structures (g,ξ) depends on the $p+q$* parameters $\lambda_1,\cdots,\lambda_p,c_1,\cdots,c_q$.

5. The de Rham decomposition for KTS-spaces

The main purpose of this final section is to prove

THEOREM 5.1. *Let (M,g,ξ) be a simply connected* KTS*-space. Then (M,g) is irreducible if and only if η is a contact form. In this case dim $M = n$ is necessarily odd. Further, if* rank$H = 2k < n-1$*, then M is a direct product*

$$(M,g) = (M',g') \times (M'',g'')$$

where (M',g',ξ) is an irreducible KTS*-space of dimension $2k+1$ and (M'',g'') is a symmetric space.*

To prove this result we shall need the following two lemmas. The first one is the KTS-version of Proposition 3.1 of [20] for φ-symmetric spaces and its proof is similar. So we omit it. (See also [13]).

LEMMA 5.1. *Let (M,g,ξ) be a simply connected* KTS*-space and let $\overline{\nabla}$ be the canonical connection of the isometric flow $\mathfrak{F}_\xi$. Then there is a coset representation of M in the form $M = G/G_o$ such that*

(i) *G is a connected Lie group with $G \subseteq A(M)$;*

(ii) *there is an $Ad(G_o)$-invariant decomposition $\mathfrak{g} = \mathfrak{g}_o \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G adapted to the naturally reductive homogeneous space $(G/G_o, g)$ for which $\overline{\nabla}$ is the canonical connection of the second kind.*

Next, we state

LEMMA 5.2 [19]. *Let $(M,g)=G/G_o$ be a simply connected naturally reductive space with an adapted canonical connection $\overline{\nabla}$. Suppose that the tangent space $V=T_oM$ at the origin admits an orthogonal decomposition $V=V_1\oplus V_2$ such that its torsion and curvature tensors satisfy*

$$\begin{aligned}\pi_i(\overline{K}(X,Y)) &= \overline{K}(\pi_iX,\pi_iY),\\ \pi_i(\overline{R}_{XY}Z) &= \overline{R}_{\pi_iX\pi_iY}\pi_iZ,\end{aligned}$$

for $i=1,2$ and all $X,Y,Z\in V$ where π_1 and π_2 denote the canonical projections. Then M is a direct product $(M,g)=(M_1,g_1)\times(M_2,g_2)$ with $\dim M_i=\dim V_i$, $i=1,2$. Here, the factors (M_i,g_i) are again naturally reductive.

Now, we are ready for the

PROOF OF THEOREM 5.1. From Lemma 5.1 it follows that $(M,g)=G/G_o$ is a naturally reductive homogeneous space with G-invariant vector field ξ and the canonical connection $\overline{\nabla}$ of the flow $\mathfrak{F}_\xi$ as canonical connection.

First, we suppose that the one-form η is not a contact form and hence, that rank $H=2k<n-1$. We shall prove that (M,g) satisfies the conditions of Lemma 5.2. Since H is skew-symmetric $V=T_oM$ admits an orthonormal basis $\{X_1,\cdots,X_{2k},X_{2k+1},\cdots,X_{n-1},\xi_o\}$ and there exist real numbers $\lambda_1,\cdots,\lambda_k$ such that

$$\begin{cases} H_oX_1 = \lambda_1X_2, & H_oX_2 = -\lambda_1X_1,\\ \cdots & \\ H_oX_{2k-1} = \lambda_kX_{2k}, & H_oX_{2k} = -\lambda_kX_{2k-1},\\ H_oX_{2k+1} = \cdots = H_oX_{n-1} = H_o\xi_o = 0. & \end{cases}$$

Now, put $V_1=\operatorname{span}\{X_1,\cdots,X_{2k},\xi_o\}$, $V_2=\operatorname{span}\{X_{2k+1},\cdots,X_{n-1}\}$. Since $\overline{K}=-2T$ and with the expression for T given in Section 2, we get that $\overline{K}_o(X_a,X_b)$ and $\overline{K}_o(X_a,\xi_o)$, $1\le a,\ b\le 2k$, belong to V_1. For $2k+1\le h,\ l\le n-1$ we get

$$\overline{K}_o(X_h,X_l)=\overline{K}_o(X_h,X_a)=\overline{K}_o(X_h,\xi_o)=0.$$

Hence,

$$\pi_i(\overline{K}_o(X,Y))=\overline{K}_o(\pi_iX,\pi_iY)$$

for all $X,Y\in V$. Further, for $1\le a,b,c\le 2k$ we get that $\overline{R}_{oX_aX_b}X_c\in V_1$. Indeed, let $U\in V_1$ such that $X_c=H_oU$. Since $\overline{R}_{X_aX_b}\cdot\overline{K}=0$ we obtain

$$\overline{R}_{oX_aX_b}X_c=\frac{1}{2}\overline{R}_{oX_aX_b}\overline{K}_o(U,\xi_o)=\frac{1}{2}\overline{K}_o(\overline{R}_{oX_aX_b}U,\xi_o)=H_o\overline{R}_{oX_aX_b}U$$

Moreover,

$$\overline{R}_{oX_iX_j}\xi_o=\overline{R}_{o\xi_oX_i}X_j=0$$

for $1\le i,j\le n-1$. Next, we show that $\overline{R}_{oX_aX_h}=0$ for $1\le a\le 2k$, $2k+1\le h\le n-1$. To see this, put $X_a=HU$, $U\in V_1$. Then, using the second Bianchi identity for $\overline{\nabla}$, we get

$$\overline{R}_{oX_aX_h}=\frac{1}{2}\overline{R}_{o\overline{K}_o(U,\xi_o)X_h}=\frac{1}{2}\left\{\overline{R}_{o\overline{K}_o(X_h,\xi_o)U}+\overline{R}_{o\overline{K}_o(U,X_h)\xi_o}\right\}=0.$$

So, if $2k+1\le l,\ m\le n-1$, we have $\overline{R}_{oX_hX_l}X_m\in V_2$. All this then yields

$$\pi_i(\overline{R}_{XY}Z)=\overline{R}_{\pi_iX\pi_iY}\pi_iZ.$$

Thus, from Lemma 5.2, we conclude that (M,g) is a direct product $(M',g')\times(M'',g'')$. Moreover, if R' denotes the curvature tensor of (M',g'), we get

$$R'(X,Y,X,\xi)=R(X,Y,X,\xi)=0$$

for all horizontal vector X,Y on M'. Moreover, η is a contact form on M' because $\operatorname{rank}H=\operatorname{rank}H_{|M'}=2k=\dim M'-1$. So, using Proposition 3.1, we obtain that (M',g',ξ) is a contact KTS-space. On the other hand, since

$$\overline{K}_o(X_h,X_l)=0,\quad 2k+1\le h,\ l\le n-1,$$

we get that (M'',g'') is a symmetric space [19, (12)].

The converse follows at once from Proposition 3.2. ■

COROLLARY 5.1. *Let (M,g,ξ) be a simply connected* KTS*-space. Then we have the following de Rham decomposition:*

$$M=M_0\times M_1\times\cdots\times M_k\times M_{k+1}$$

where M_0 is a Euclidean space, $M_1,\cdots,M_k$ are irreducible symmetric spaces and M_{k+1} is a contact KTS*-space. Moreover, M fibers over the symmetric space*

$$M'=M_0\times M_1\times\cdots\times M_k\times M'_{k+1}$$

where M'_{k+1} is the simply connected Hermitian symmetric space M_{k+1}/ξ.

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